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CARTAN CALCULUS ON THE QUANTUM SPACE \mathcal{R}_q^3

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ABSTRACT

To give a Cartan calculus on the extended quantum 3d space, the noncommutative differential calculus on the extended quantum 3d space is extended by introducing inner derivations and Lie derivatives.

1. INTRODUCTION

The noncommutative differential geometry of quantum groups was introduced by Woronowicz [11,12]. In this approach the differential calculus on the group is deduced from the properties of the group and it involves functions on the group, differentials, differential forms and derivatives. The other approach, initiated by Wess and Zumino [10], followed Manin's emphasis [5] on the quantum spaces as the primary objects. Differential forms are defined in terms of noncommuting coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces.

The differential calculus on the quantum 3d space similarly involves functions on the 3d space, differentials, differential forms and derivatives. The exterior derivative is a linear operator d acting on k -forms and producing $(k+1)$ -forms, such that for scalar functions (0-forms) f and g we have

$$\begin{aligned} d(1) &= 0, \\ d(fg) &= (df)g + (-1)^{\deg(f)} f(dg) \end{aligned}$$

where $\deg(f) = 0$ for even variables and $\deg(f) = 1$ for odd variables, and for a k -form ω_1 and any form ω_2

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (d\omega_2).$$

A fundamental property of the exterior derivative d is

$$d \wedge d = : d^2 = 0.$$

There is a relationship of the exterior derivative with the Lie derivative and to describe this relation, we introduce a new operator: the inner derivation. Hence the differential calculus on the quantum 3d space can be extended into a large calculus. We call this new calculus the Cartan calculus. The connection of the inner derivation denoted by i_a and the Lie derivative denoted by \mathcal{L}_a is given by the Cartan formula:

$$\mathcal{L}_a = i_a \circ d + d \circ i_a.$$

This and other formulae are explained in Ref. 6-8. We now shall give a brief overview without much discussion.

Let us begin with some information about the inner derivations. Generally, for a smooth vector field X on a manifold the inner derivation, denoted by \mathbf{i}_X , is a linear operator which maps k -forms to $(k - 1)$ -forms. If we define the inner derivation \mathbf{i}_X on the set of all differential forms on a manifold, we know that \mathbf{i}_X is an antiderivation of degree -1 :

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta)$$

where α and β are both differential forms. The inner derivation \mathbf{i}_X acts on 0- and 1-forms as follows:

$$\begin{aligned} \mathbf{i}_X(f) &= 0, \\ \mathbf{i}_X(df) &= X(f). \end{aligned}$$

We know, from the classical differential geometry, that the Lie derivative \mathcal{L} can be defined as a linear map from the exterior algebra into itself which takes k -forms to k -forms. For a 0-form, that is, an ordinary function f , the Lie derivative is just the contraction of the exterior derivative with the vector field X :

$$\mathcal{L}_X f = \mathbf{i}_X df.$$

For a general differential form, the Lie derivative is likewise a contraction, taking into account the variation in X :

$$\mathcal{L}_X \alpha = \mathbf{i}_X d\alpha + d(\mathbf{i}_X \alpha).$$

The Lie derivative has the following properties. If $\mathcal{F}(M)$ is the algebra of functions defined on the manifold M then

$$\mathcal{L}_X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$

is a derivation on the algebra $\mathcal{F}(M)$:

$$\begin{aligned} \mathcal{L}_X(af + bg) &= a(\mathcal{L}_X f) + b(\mathcal{L}_X g), \\ \mathcal{L}_X(fg) &= (\mathcal{L}_X f)g + f(\mathcal{L}_X g), \end{aligned}$$

where a and b real numbers.

The Lie derivative is a derivation on $\mathcal{F}(M) \times \mathcal{V}(M)$ where $\mathcal{V}(M)$ is the set of vector fields on M :

$$\mathcal{L}_{X_1}(fX_2) = (\mathcal{L}_{X_1}f)X_2 + f(\mathcal{L}_{X_1}X_2).$$

The Lie derivative also has an important property when acting on differential forms. If α and β are two differential forms on M then

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathcal{L}_X\beta)$$

where α is a k -form.

The extended calculus on the quantum plane was introduced in Ref. 3 using the approach of Ref. 6. In this work we explicitly set up the Cartan calculus on the quantum 3d space using approach of Ref 1.

2. REVIEW OF SOME STRUCTURES ON \mathcal{R}_q^3

In this section we give some information on the Hopf algebra structures of the quantum 3d space and its differential calculus [2] which we shall use in order to establish our notions.

2.1 The algebra of polynomials on the quantum 3d space

The quantum three dimensional space is defined as an associative algebra generated by three noncommuting coordinates x, y and z with three quadratic relations

$$\begin{aligned} xy &= qyx, \\ yz &= qzy, \\ xz &= qzx, \end{aligned}$$

where q is a non-zero complex number. This associative algebra over the complex number, \mathbb{C} , is known as the algebra of polynomials over the quantum three dimensional space and we shall denote it by \mathcal{R}_q^3 . In the limit $q \longrightarrow 1$, this algebra is commutative and can be considered as the algebra of polynomials $\mathbb{C}[x, y, z]$ over

the usual three dimensional space, where x , y and z are the three coordinate functions. We denote the unital extension of \mathcal{R}_q^3 by \mathcal{A} , i.e. it is obtained by adding a unit element.

2.2 The Hopf algebra structure on \mathcal{A}

One extends the algebra \mathcal{A} by including inverse of x which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

The definitions of a coproduct, a counit and a coinverse on the algebra \mathcal{A} as follows [2]:

(1) The C-algebra homomorphism (coproduct) $\Delta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\begin{aligned}\Delta_{\mathcal{A}}(x) &= x \otimes x, \\ \Delta_{\mathcal{A}}(y) &= x \otimes y + y \otimes x, \\ \Delta_{\mathcal{A}}(z) &= z \otimes 1 + 1 \otimes z,\end{aligned}$$

which is coassociative:

$$(\Delta_{\mathcal{A}} \otimes \text{id}) \circ \Delta_{\mathcal{A}} = (\text{id} \otimes \Delta_{\mathcal{A}}) \circ \Delta_{\mathcal{A}}$$

where id denotes the identity map on \mathcal{A} .

(2) The C-algebra homomorphism (counit) $\epsilon_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbb{C}$ is given by

$$\begin{aligned}\epsilon_{\mathcal{A}}(x) &= 1, \\ \epsilon_{\mathcal{A}}(y) &= 0, \\ \epsilon_{\mathcal{A}}(z) &= 0.\end{aligned}$$

The counit $\epsilon_{\mathcal{A}}$ has the property

$$\mu \circ (\epsilon_{\mathcal{A}} \otimes \text{id}) \circ \Delta_{\mathcal{A}} = \mu' \circ (\text{id} \otimes \epsilon_{\mathcal{A}}) \circ \Delta_{\mathcal{A}}$$

where $\mu : \mathbb{C} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and $\mu' : \mathcal{A} \otimes \mathbb{C} \longrightarrow \mathcal{A}$ are the canonical isomorphisms, defined by

$$\mu(k \otimes u) = ku = \mu'(u \otimes k), \quad \forall u \in \mathcal{A}, \quad \forall k \in \mathbb{C}.$$

(3) The C-algebra antihomomorphism (coinverse) $S_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\begin{aligned} S_{\mathcal{A}}(x) &= x^{-1}, \\ S_{\mathcal{A}}(y) &= -x^{-1}yx^{-1}, \\ S_{\mathcal{A}}(z) &= -z. \end{aligned}$$

The coinverse S satisfies

$$m \circ (S_{\mathcal{A}} \otimes \text{id}) \circ \Delta_{\mathcal{A}} = \epsilon_{\mathcal{A}} = m \circ (\text{id} \otimes S_{\mathcal{A}}) \circ \Delta_{\mathcal{A}}$$

where m stands for the algebra product $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$.

The coproduct, counit and coinverse which are specified above supply the algebra \mathcal{A} with a Hopf algebra structure.

2.3 Differential algebra

We first note that the properties of the exterior differential \mathbf{d} . The exterior differential \mathbf{d} is an operator which gives the mapping from the generators of \mathcal{A} to the differentials:

$$\mathbf{d} : u \longrightarrow \mathbf{d}u, \quad u \in \{x, y, z\}.$$

We demand that the exterior differential \mathbf{d} has to satisfy two properties: the nilpotency

$$\mathbf{d}^2 = 0$$

and the Leibniz rule

$$\mathbf{d}(fg) = (\mathbf{d}f)g + (-1)^{\deg(f)} f(\mathbf{d}g).$$

A deformed differential calculus on the quantum 3d space is as follows:

the commutation relations with the coordinates of differentials

$$\begin{aligned} x \, \mathbf{d}x &= \mathbf{d}x \, x, & x \, \mathbf{d}y &= q \mathbf{d}y \, x, & x \, \mathbf{d}z &= q \mathbf{d}z \, x, \\ y \, \mathbf{d}x &= q^{-1} \mathbf{d}x \, y, & y \, \mathbf{d}y &= \mathbf{d}y \, y, & y \, \mathbf{d}z &= q \mathbf{d}z \, y, \\ z \, \mathbf{d}x &= q^{-1} \mathbf{d}x \, z, & z \, \mathbf{d}y &= q^{-1} \mathbf{d}y \, z, & z \, \mathbf{d}z &= \mathbf{d}z \, z. \end{aligned}$$

This algebra is denoted by Γ_1 .

The commutation relations between the differentials

$$\begin{aligned} dx \wedge dx &= 0, & dy \wedge dy &= 0, & dz \wedge dz &= 0. \\ dx \wedge dy &= -q dy \wedge dx, \\ dy \wedge dz &= -q dz \wedge dy, \\ dx \wedge dz &= -q dz \wedge dx. \end{aligned}$$

This algebra is denoted by Γ_2 .

A differential algebra on an associative algebra \mathcal{A} is a graded associative algebra Γ equipped with an operator d that has the above properties. Furthermore, the algebra Γ has to be generated by $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_0 is isomorphic to \mathcal{A} . Let Γ be the quotient algebra of the free associative algebra on the set $\{x, y, z, dx, dy, dz\}$ modulo the ideal J that is generated by the relations of \mathcal{R}_q^3 , Γ_1 and Γ_2 .

To proceed, one can obtain the relations of the coordinates with their partial derivatives using the expression

$$df = (dx \partial_x + dy \partial_y + dz \partial_z)f.$$

Consequently one has

$$\begin{aligned} \partial_x x &= 1 + x \partial_x, & \partial_x y &= q^{-1} y \partial_x, & \partial_x z &= q^{-1} z \partial_x, \\ \partial_y x &= qx \partial_y, & \partial_y y &= 1 + y \partial_y, & \partial_y z &= q^{-1} z \partial_y, \\ \partial_z x &= qx \partial_z, & \partial_z y &= qy \partial_z, & \partial_z z &= 1 + z \partial_z. \end{aligned}$$

Using the fact that $d^2 = 0$, one finds

$$\begin{aligned} \partial_x \partial_y &= q \partial_y \partial_x, \\ \partial_x \partial_z &= q \partial_z \partial_x, \\ \partial_y \partial_z &= q \partial_z \partial_y. \end{aligned}$$

The relations between partial derivatives and differentials are found as

$$\begin{aligned} \partial_x dx &= dx \partial_x, & \partial_x dy &= q^{-1} dy \partial_x, & \partial_x dz &= q^{-1} dz \partial_x, \\ \partial_y dx &= q dx \partial_y, & \partial_y dy &= dy \partial_y, & \partial_y dz &= q^{-1} dz \partial_y, \\ \partial_z dx &= q dx \partial_z, & \partial_z dy &= q dy \partial_z, & \partial_z dz &= dz \partial_z. \end{aligned}$$

We can define three one-forms using the generators of \mathcal{A} . If we call them ω_x , ω_y and ω_z then one can define them as follows:

$$\begin{aligned}\omega_x &= dx x^{-1}, \\ \omega_y &= dy x^{-1} - dx x^{-1} y x^{-1}, \\ \omega_z &= dz.\end{aligned}$$

We denote the algebra of forms generated by three elements ω_x , ω_y and ω_z by Ω . The generators of the algebra Ω with the generators of \mathcal{A} satisfy the following rules

$$\begin{aligned}x\omega_x &= \omega_x x, & x\omega_y &= q\omega_y x, & x\omega_z &= q\omega_z x, \\ y\omega_x &= \omega_x y, & y\omega_y &= q\omega_y y, & y\omega_z &= q\omega_z y, \\ z\omega_x &= \omega_x z, & z\omega_y &= \omega_y z, & z\omega_z &= \omega_z z.\end{aligned}$$

The commutation rules of the generators of Ω are

$$\begin{aligned}\omega_x \wedge \omega_x &= 0, & \omega_y \wedge \omega_y &= 0, & \omega_z \wedge \omega_z &= 0, \\ \omega_x \wedge \omega_y &= -\omega_y \wedge \omega_x, \\ \omega_y \wedge \omega_z &= -\omega_z \wedge \omega_y, \\ \omega_x \wedge \omega_z &= -\omega_z \wedge \omega_x.\end{aligned}$$

The algebra Ω is a graded Hopf algebra [2].

2.4 Lie algebra

The commutation relations of Cartan-Maurer forms allow us to construct the algebra of the generators. In order to obtain the quantum Lie algebra of the algebra generators we first write the Cartan-Maurer forms as

$$\begin{aligned}dx &= \omega_x x, \\ dy &= \omega_x y + \omega_y x, \\ dz &= \omega_z.\end{aligned}$$

The differential d can then be expressed in the form

$$df = (\omega_x T_x + \omega_y T_y + \omega_z T_z) f.$$

Here T_x , T_y and T_z are the quantum Lie algebra generators. Considering an arbitrary function f of the coordinates of the quantum 3d space and using that $d^2 = 0$, we find the following commutation relations for the (undeformed) Lie algebra [2]:

$$\begin{aligned}[T_x, T_y] &= 0, \\ [T_x, T_z] &= 0, \\ [T_y, T_z] &= 0.\end{aligned}$$

The commutation relations between the generators of algebra and the coordinates are

$$\begin{aligned}T_x x &= x + x T_x, & T_x y &= y + y T_x, & T_x z &= z T_x, \\ T_y x &= qx T_y, & T_y y &= x + qy T_y, & T_y z &= z T_y, \\ T_z x &= qx T_z, & T_z y &= qy T_z, & T_z z &= 1 + z T_z.\end{aligned}$$

The (quantum) Lie algebra generators can be expressed in terms of the generators of the quantum 3d space and partial differentials:

$$\begin{aligned}T_x &\equiv x\partial_x + y\partial_y, \\ T_y &\equiv x\partial_y, \\ T_z &\equiv \partial_z.\end{aligned}$$

The commutation relations of the Lie algebra generators T_x , T_y and T_z with the differentials are following

$$\begin{aligned}T_x dx &= dx T_x, & T_x dy &= dy T_x, & T_x dz &= dz T_x, \\ T_y dx &= qdx T_y, & T_y dy &= qdy T_y, & T_y dz &= dz T_y, \\ T_z dx &= qdx T_z, & T_z dy &= qdy T_z, & T_z dz &= dz T_z.\end{aligned}$$

The commutation rules of the Lie algebra generators with one-forms as follows

$$\begin{aligned}T_x \omega_x &= \omega_x T_x - \omega_x, & T_x \omega_y &= \omega_y T_x - \omega_y, & T_x \omega_z &= \omega_z T_x, \\ T_y \omega_x &= \omega_x T_y, & T_y \omega_y &= \omega_y T_y - \omega_x, & T_y \omega_z &= \omega_z T_y, \\ T_z \omega_x &= \omega_x T_z, & T_z \omega_y &= \omega_y T_z, & T_z \omega_z &= \omega_z T_z.\end{aligned}$$

The Hopf algebra structure of the Lie algebra generators is given by

$$\begin{aligned}
\Delta(T_x) &= T_x \otimes \mathbf{1} + \mathbf{1} \otimes T_x, \\
\Delta(T_y) &= T_y \otimes \mathbf{1} + q^{T_x} \otimes T_y, \\
\Delta(T_z) &= T_z \otimes \mathbf{1} + q^{T_x} \otimes T_z, \\
\epsilon(T_x) &= 0, \quad \epsilon(T_y) = 0, \quad \epsilon(T_z) = 0, \\
S(T_x) &= -T_x, \quad S(T_y) = -q^{-T_x} T_y, \quad S(T_z) = -q^{-T_x} T_z.
\end{aligned}$$

2.5 The dual of the Hopf algebra \mathcal{A}

In this section, in order to obtain the dual of the Hopf algebra \mathcal{A} defined in section 2, we shall use the method of Refs. 4 and 9.

A pairing between two vector spaces \mathcal{U} and \mathcal{A} is a bilinear mapping

$$\langle, \rangle: \mathcal{U} \times \mathcal{A} \longrightarrow \mathcal{C}, \quad (u, a) \mapsto \langle u, a \rangle.$$

We say that the pairing is non-degenerate if

$$\langle u, a \rangle = 0 \quad (\forall a \in \mathcal{A}) \implies u = 0$$

and

$$\langle u, a \rangle = 0 \quad (\forall u \in \mathcal{U}) \implies a = 0.$$

Such a pairing can be extended to a pairing of $\mathcal{U} \otimes \mathcal{U}$ and $\mathcal{A} \otimes \mathcal{A}$ by

$$\langle u \otimes v, a \otimes b \rangle = \langle u, a \rangle \langle v, b \rangle.$$

Given bialgebras \mathcal{U} and \mathcal{A} and a non-degenerate pairing

$$\langle, \rangle: \mathcal{U} \times \mathcal{A} \longrightarrow \mathcal{C} \quad (u, a) \mapsto \langle u, a \rangle \quad \forall u \in \mathcal{U} \quad \forall a \in \mathcal{A}$$

we say that the bilinear form realizes a duality between \mathcal{U} and \mathcal{A} , or that the bialgebras \mathcal{U} and \mathcal{A} are in duality, if we have

$$\begin{aligned}
\langle uv, a \rangle &= \langle u \otimes v, \Delta_{\mathcal{A}}(a) \rangle, \\
\langle u, ab \rangle &= \langle \Delta_{\mathcal{U}}(u), a \otimes b \rangle, \\
\langle 1_{\mathcal{U}}, a \rangle &= \epsilon_{\mathcal{A}}(a),
\end{aligned}$$

and

$$\langle u, 1_{\mathcal{A}} \rangle = \epsilon_{\mathcal{U}}(u)$$

for all $u, v \in \mathcal{U}$ and $a, b \in \mathcal{A}$.

If, in addition, \mathcal{U} and \mathcal{A} are Hopf algebras with coinverse κ , then they are said to be in duality if the underlying bialgebras are in duality and if, moreover, we have

$$\langle S_{\mathcal{U}}(u), a \rangle = \langle u, S_{\mathcal{A}}(a) \rangle \quad \forall u \in \mathcal{U} \quad a \in \mathcal{A}.$$

It is enough to define the pairing between the generating elements of the two algebras. Pairing for any other elements of \mathcal{U} and \mathcal{A} follows from above relations and the bilinear form inherited by the tensor product. For example, for

$$\Delta_{\mathcal{U}}(u) = \sum_k u'_k \otimes u''_k,$$

we have

$$\langle u, ab \rangle = \langle \Delta_{\mathcal{U}}(u), a \otimes b \rangle = \sum_k \langle u'_k, a \rangle \langle u''_k, b \rangle$$

As a Hopf algebra \mathcal{A} is generated by the elements x, y and z , and a basis is given by all monomials of the form

$$f = x^k y^l z^m$$

where $k, l, m \in \mathbb{Z}_+$. Let us denote the dual algebra by \mathcal{U}_q and its generating elements by A and B .

The pairing is defined through the tangent vectors as follows

$$\begin{aligned} \langle X, f \rangle &= k \delta_{l,0} \delta_{m,0}, \\ \langle Y, f \rangle &= \delta_{l,1} \delta_{m,0}, \\ \langle Z, f \rangle &= \delta_{l,0} \delta_{m,1}. \end{aligned}$$

We also have

$$\langle 1_{\mathcal{U}}, f \rangle = \epsilon_{\mathcal{A}}(f) = \delta_{k,0}.$$

Using the defining relations one gets

$$\langle XY, f \rangle = \delta_{l,1} \delta_{m,0}$$

and

$$\langle YX, f \rangle = \delta_{l,1} \delta_{m,0}$$

where differentiation is from the right as this is most suitable for differentiation in this basis. Thus one obtains one of the commutation relations in the algebra \mathcal{U}_q dual to \mathcal{A} as:

$$XY = YX.$$

Similarly, one has

$$XZ = ZX,$$

$$YZ = ZY.$$

The Hopf algebra structure of this algebra can be deduced by using the duality. The coproduct of the elements of the dual algebra is given by

$$\Delta_{\mathcal{U}}(X) = X \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes X,$$

$$\Delta_{\mathcal{U}}(Y) = Y \otimes q^{-X} + 1_{\mathcal{U}} \otimes Y,$$

$$\Delta_{\mathcal{U}}(Z) = Z \otimes q^{-X} + 1_{\mathcal{U}} \otimes Z.$$

The counity is given by

$$\epsilon_{\mathcal{U}}(X) = 0, \quad \epsilon_{\mathcal{U}}(Y) = 0, \quad \epsilon_{\mathcal{U}}(Z) = 0.$$

The coinverse is given as

$$S_{\mathcal{U}}(X) = -X, \quad S_{\mathcal{U}}(Y) = -Yq^X, \quad S_{\mathcal{U}}(Z) = -Zq^X.$$

We can now transform this algebra to the form obtained in section 5 by making the following identities:

$$T_x \equiv X, \quad T_y \equiv q^{X/2} Y q^{X/2}, \quad T_z \equiv q^{X/2} Z q^{X/2}$$

which are consistent with the commutation relation and the Hopf structures.

3. EXTENDED CALCULUS ON THE QUANTUM 3D SPACE

A Lie derivative is a derivation on the algebra of tensor fields over a manifold. The Lie derivative should be defined three ways: on scalar functions, vector fields and tensors.

The Lie derivative can also be defined on differential forms. In this case, it is closely related to the exterior derivative. The exterior derivative and the Lie derivative are set to cover the idea of a derivative in different ways. These differences can be hasped together by introducing the idea of an antiderivation which is called an inner derivation.

3.1 Inner derivations

In order to obtain the commutation rules of the coordinates with inner derivations, we shall use the approach of Ref. 1. Similarly other relations can also obtain. Consequently, we have the following commutation relations:

- the commutation relations of the inner derivations with x , y and z

$$\begin{aligned} \mathbf{i}_x x &= x \mathbf{i}_x, & \mathbf{i}_x y &= q^{-1} y \mathbf{i}_x, & \mathbf{i}_x z &= q^{-1} z \mathbf{i}_x, \\ \mathbf{i}_y x &= q x \mathbf{i}_y, & \mathbf{i}_y y &= y \mathbf{i}_y, & \mathbf{i}_y z &= q^{-1} z \mathbf{i}_y, \\ \mathbf{i}_z x &= q x \mathbf{i}_z, & \mathbf{i}_z y &= q y \mathbf{i}_z, & \mathbf{i}_z z &= z \mathbf{i}_z. \end{aligned}$$

- the relations of the inner derivations with the partial derivatives ∂_x , ∂_y and ∂_z

$$\begin{aligned} \mathbf{i}_x \partial_x &= \partial_x \mathbf{i}_x, & \mathbf{i}_x \partial_y &= q \partial_y \mathbf{i}_x, & \mathbf{i}_x \partial_z &= q \partial_z \mathbf{i}_x, \\ \mathbf{i}_y \partial_x &= q^{-1} \partial_x \mathbf{i}_y, & \mathbf{i}_y \partial_y &= \partial_y \mathbf{i}_y, & \mathbf{i}_y \partial_z &= q \partial_z \mathbf{i}_y, \\ \mathbf{i}_z \partial_x &= q^{-1} \partial_x \mathbf{i}_z, & \mathbf{i}_z \partial_y &= q^{-1} \partial_y \mathbf{i}_z, & \mathbf{i}_z \partial_z &= \partial_z \mathbf{i}_z. \end{aligned}$$

- the commutation relations between the differentials and the inner derivations

$$\begin{aligned} \mathbf{i}_x \wedge dx &= 1 - dx \wedge \mathbf{i}_x, & \mathbf{i}_x \wedge dy &= -q^{-1} dy \wedge \mathbf{i}_x, \\ \mathbf{i}_y \wedge dx &= -q dx \wedge \mathbf{i}_y, & \mathbf{i}_y \wedge dy &= 1 - dy \wedge \mathbf{i}_y, \\ \mathbf{i}_z \wedge dx &= -q dx \wedge \mathbf{i}_z, & \mathbf{i}_z \wedge dy &= -q dy \wedge \mathbf{i}_z, \end{aligned}$$

$$\begin{aligned}
\mathbf{i}_x \wedge \mathrm{d}z &= -q^{-1} \mathrm{d}z \wedge \mathbf{i}_x, & \mathbf{i}_y \wedge \mathrm{d}z &= -q^{-1} \mathrm{d}z \wedge \mathbf{i}_y, \\
\mathbf{i}_z \wedge \mathrm{d}z &= 1 - \mathrm{d}z \wedge \mathbf{i}_z.
\end{aligned}$$

3.2 Lie derivations

In this section we find the commutation rules of the Lie derivatives with functions, *i.e.* the elements of the algebra \mathcal{A} , their differentials, etc., using the approach of [1] as follows:

- the relations between the Lie derivatives and the elements of \mathcal{A}

$$\begin{aligned}
\mathcal{L}_x x &= 1 + x \mathcal{L}_x, & \mathcal{L}_x y &= q^{-1} y \mathcal{L}_x, & \mathcal{L}_x z &= q^{-1} z \mathcal{L}_x, \\
\mathcal{L}_y x &= q x \mathcal{L}_y, & \mathcal{L}_y y &= 1 + y \mathcal{L}_y, & \mathcal{L}_y z &= q^{-1} z \mathcal{L}_y, \\
\mathcal{L}_z x &= q x \mathcal{L}_z, & \mathcal{L}_z y &= q y \mathcal{L}_z, & \mathcal{L}_z z &= 1 + z \mathcal{L}_z.
\end{aligned}$$

- The relations of the Lie derivatives with the differentials

$$\begin{aligned}
\mathcal{L}_x \mathrm{d}x &= \mathrm{d}x \mathcal{L}_x, & \mathcal{L}_x \mathrm{d}y &= q^{-1} \mathrm{d}y \mathcal{L}_x, & \mathcal{L}_x \mathrm{d}z &= q^{-1} \mathrm{d}z \mathcal{L}_x, \\
\mathcal{L}_y \mathrm{d}x &= q \mathrm{d}x \mathcal{L}_y, & \mathcal{L}_y \mathrm{d}y &= \mathrm{d}y \mathcal{L}_y, & \mathcal{L}_y \mathrm{d}z &= q^{-1} \mathrm{d}z \mathcal{L}_y, \\
\mathcal{L}_z \mathrm{d}x &= q \mathrm{d}x \mathcal{L}_z, & \mathcal{L}_z \mathrm{d}y &= q \mathrm{d}y \mathcal{L}_z, & \mathcal{L}_z \mathrm{d}z &= \mathrm{d}z \mathcal{L}_z.
\end{aligned}$$

Other commutation relations can be similarly obtained. To complete the description of the above scheme, we get below the remaining commutation relations as follows:

- the Lie derivatives and partial derivatives

$$\begin{aligned}
\mathcal{L}_x \partial_x &= \partial_x \mathcal{L}_x, & \mathcal{L}_x \partial_y &= q \partial_y \mathcal{L}_x, & \mathcal{L}_x \partial_z &= q \partial_z \mathcal{L}_x, \\
\mathcal{L}_y \partial_x &= q^{-1} \partial_x \mathcal{L}_y, & \mathcal{L}_y \partial_y &= \partial_y \mathcal{L}_y, & \mathcal{L}_y \partial_z &= q \partial_z \mathcal{L}_y, \\
\mathcal{L}_z \partial_x &= q^{-1} \partial_x \mathcal{L}_z, & \mathcal{L}_z \partial_y &= q^{-1} \partial_y \mathcal{L}_z, & \mathcal{L}_z \partial_z &= \partial_z \mathcal{L}_z.
\end{aligned}$$

- the inner derivations

$$\begin{aligned}
\mathbf{i}_x \wedge \mathbf{i}_y &= -q \mathbf{i}_y \wedge \mathbf{i}_x, \\
\mathbf{i}_x \wedge \mathbf{i}_z &= -q \mathbf{i}_z \wedge \mathbf{i}_x, \\
\mathbf{i}_y \wedge \mathbf{i}_z &= -q \mathbf{i}_z \wedge \mathbf{i}_y.
\end{aligned}$$

- the Lie derivatives and the inner derivations

$$\begin{aligned}
\mathcal{L}_x \mathbf{i}_x &= \mathbf{i}_x \mathcal{L}_x, & \mathcal{L}_x \mathbf{i}_y &= q \mathbf{i}_y \mathcal{L}_x, & \mathcal{L}_x \mathbf{i}_z &= q \mathbf{i}_z \mathcal{L}_x, \\
\mathcal{L}_y \mathbf{i}_x &= q^{-1} \mathbf{i}_x \mathcal{L}_y, & \mathcal{L}_y \mathbf{i}_y &= \mathbf{i}_y \mathcal{L}_y, & \mathcal{L}_y \mathbf{i}_z &= q \mathbf{i}_z \mathcal{L}_y, \\
\mathcal{L}_z \mathbf{i}_x &= q^{-1} \mathbf{i}_x \mathcal{L}_z, & \mathcal{L}_z \mathbf{i}_y &= q^{-1} \mathbf{i}_y \mathcal{L}_z, & \mathcal{L}_z \mathbf{i}_z &= \mathbf{i}_z \mathcal{L}_z.
\end{aligned}$$

- the Lie derivatives

$$\begin{aligned}
\mathcal{L}_x \mathcal{L}_y &= q \mathcal{L}_y \mathcal{L}_x, \\
\mathcal{L}_x \mathcal{L}_z &= q \mathcal{L}_z \mathcal{L}_x, \\
\mathcal{L}_y \mathcal{L}_z &= q \mathcal{L}_z \mathcal{L}_y.
\end{aligned}$$

Note that the Lie derivatives can be written as follows:

$$\begin{aligned}
\mathcal{L}_x &= x^{-1} T_x - x^{-1} y x^{-1} T_y, \\
\mathcal{L}_y &= x^{-1} T_y, \\
\mathcal{L}_z &= T_z.
\end{aligned}$$

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